The Yamabe equation on manifolds of bounded geometry

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Abstract

We study the Yamabe problem on open manifolds of bounded geometry and show that under suitable assumptions there exist Yamabe metrics, i.e. conformal metrics of constant scalar curvature. For that, we use weighted Sobolev embeddings.

1 Introduction

In 1960 Yamabe considered the following problem that became famous as the Yamabe problem:

Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. Does there exist a Riemannian metric $\overline{g}$ conformal to g that has constant scalar curvature?

This was answered affirmatively by Aubin [5], Schoen [13] and Trudinger [18].
The question can be reformulated in terms of positive solutions of the nonlinear elliptic differential equation:

$$c u^{\frac{n+2}{n-2}} = L_g u \quad \|u\|_{p_{crit}} = 1 \tag{1}$$

where c is a constant, $L_g = a_n \Delta_g + \mathrm{scal}_g$ with $a_n = 4\frac{n-1}{n-2}$ is the conformal Laplacian and scal_g the scalar curvature. We denote $\|u\|_p := \|u\|_{L^p(g)}$ and set $p_{crit} = \frac{2n}{n-2}$. In the following we will omit the index referring to the metric, e.g. $L = L_g$.
If a positive solution u exists, then the conformal metric $\overline{g} = u^{\frac{4}{n-2}} g$ has constant scalar curvature. Moreover, solutions of (1) can be characterized as critical points of the Yamabe functional

$$Q_g(v) = \frac{\int_M v L_g v \mathrm{dvol}_g}{\|v\|^2_{p_{crit}}}.$$

The infimum of the Yamabe functional $Q(M, g) = \inf\{Q_g(v) \mid v \in C_c^\infty(M)\}$ is called the *Yamabe invariant* of (M, g), where $C_c^\infty(M)$ denotes the set of compactly supported real valued functions on M.

Since we take the infimum over all functions with compact support, the definition of the Yamabe invariant can also be used for noncompact manifolds.
What is often referred to as the noncompact Yamabe problem is the question: Let (M, g) be a complete Riemannian manifold of dimension $n \geq 3$. Does there exist a complete metric $\overline{g}$ conformal to g that has constant scalar curvature?

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The simplest counterexample is the standard Euclidean space. But in [9] it was shown that by deleting finitely many points on a compact manifold one can always construct such counterexamples.
Another way to consider a noncompact version of the Yamabe problem is to ask for a positive solution $u \in H_1^2 \cap L^p$ of (1) on a noncompact complete manifold that minimizes the Yamabe functional. Here, $H_1^2 = H_1^2(g)$ is the completion of $C_c^\infty(M)$ with respect to the norm $\|v\|_{H_1^2(g)} := \|\nabla v\|_{L^2(g)} + \|v\|_{L^2(g)}$. The corresponding conformal metric $u^{\frac{4}{n-2}}g$ will have constant scalar curvature but will be in general not complete.

In this paper, we want to examine the existence of solutions of the Euler-Lagrange equation that minimize the Yamabe functional, i.e. we consider the second version of the noncompact Yamabe problem described above.
In [10], this problem was studied for positive scalar curvature. In the proof, Aubin's inequality is used which was proofed in [4, Thm. 9] for compact manifolds. Unfortunately, this inequality is not true for an arbitrary open manifold, but the proof of Aubin's inequality on compact manifolds carries over to manifolds with bounded geometry. Thus, in the assumptions of [10, Thm. 1] bounded geometry should be inserted to make the proof work. In the following, we want to extend this result by relaxing the assumptions on the scalar curvature. Instead of assuming positive scalar curvature, we will assume that $\mu(M,g)$, the infimum of the L^2-spectrum of the conformal Laplacian w.r.t. the complete metric g, i.e.

$$\mu(M,g) = \inf\left\{ \int_M vLv \,\mathrm{dvol}_g \;\middle|\; v \in C_c^\infty(M),\ \|v\|_2 = 1 \right\}$$

is positive.

Theorem 1. *Let (M,g) be a connected manifold of bounded geometry with $\overline{Q(M,[g])} > Q(M,[g])$. Moreover, let $\mu(M,g) > 0$. Then, there is a smooth positive solution $v \in H_1^2$ of the Euler-Lagrange equation $Lv = Q(M)v^{p_{crit}-1}$ with $\|v\|_{p_{crit}} = 1$.*

Here, $\overline{Q}$ denotes the Yamabe invariant at infinity, cf. Definition 4. Note, moreover, that $\mu > 0$ implies $Q > 0$, see Lemma 7.
Our method to prove this theorem will be different to the one in [10], where the noncompact manifold is exhausted by compact subsets. There the solutions of the corresponding problem on these subsets form a sequence and it is shown that under suitable assumptions this sequence converges to a global solution.
We will use instead weighted Sobolev embeddings and, therefore, consider a weighted Yamabe problem:

Definition 2. *Let ρ be a radial admissible weight (cf. [17, Def. 2]) with $\rho \leq 1$. The* weighted subcritical Yamabe constant *is defined as*

$$Q_p^\alpha(M,g) = \inf\left\{ \int_M vLv \,\mathrm{dvol}_g \;\middle|\; v \in C_c^\infty(M),\ \|\rho^\alpha v\|_p = 1 \right\}$$

where $\alpha \geq 0$ and $p \in [2, p_{crit})$, $p_{crit} = \frac{2n}{n-2}$. If $\alpha = 0$, we simply write Q_p.

For our purpose, it will be sufficient to think of ρ as the radial weight e^{-r} where r is smooth and near to the distance to a fixed point $z \in M$, cf. the Appendix A Remark 18.
Note that $Q = Q_{p=p_{crit}}^{\alpha=0}$.

At the end, we will show that for homogeneous manifolds with strictly positive scalar curvature one can drop the assumption on $\overline{Q}$. This was shown to the author by Akutagawa who

proved this by exhaustion of the manifold at infinity, similarly as in [2, Thm. C]. Similar methods are used in [1, Thm. 1.2], where Akutagawa compares the Yamabe constant of a manifold M with the Yamabe constant on an infinite covering of M.

In this paper, we will proceed as follows: In section 2, we shortly give some general results and the definition of the Yamabe invariant at infinity. Everything that is needed on (weighted) Sobolev embeddings can be found in Appendix A. In section 3, we will prove Theorem 1 by considering a weighted subcritical problem.
The methods developped in this paper to prove existence of solutions of the Yamabe problem on manifold with bounded geometry were adapted to prove similar results for a spinorial Yamabe-type problem for the Dirac operator. That was done in [7].
Acknowlegdement. The author thanks Kazuo Akutagawa for giving many insights to the solutions of the Yamabe problem on noncompact manifolds and showing Theorem 13 which we reproved here by our method. Furthermore, I want to thank Bernd Ammann for many enlightening discussions and hints on weighted Sobolev embeddings.

2 Preliminaries

In the rest of the paper, let (M, g) be an n-dimensional complete connected Riemannian manifold. We will focus on properties of the Yamabe invariant on manifolds. For statements on embeddings, especially on weighted Sobolev embeddings, we refer to Appendix A.

We will collect some basic properties for the Yamabe invariant on manifolds (not necessarily compact or complete but always without boundary) which we will need in the following, cf. [14].

Theorem 3. *Let $\Omega_1 \subset \Omega_2 \subset M$ be open subsets of the Riemannian manifold (M, g) equipped with the induced metric. Then $Q(\Omega_1, g) \geq Q(\Omega_2, g) \geq Q(M, g)$. Moreover,*

$$Q(M, g) \leq Q(S^n, g_{st}) = n(n-1)\omega_n^{\frac{2}{n}}$$

where ω_n is the volume of the standard sphere (S^n, g_{st}).
For any open subset $\Omega \subset S^n$ of the standard sphere, it is $Q(\Omega, g_{st}) = Q(S^n, g_{st})$. In particular, the Yamabe invariants of the standard Euclidean and hyperbolic space coincide with the one of the standard sphere.

In the sequel, we will left out the metric in the notation of Q if it is clear from context to which metric we refer to, e.g. in case of the standard sphere we just write $Q(S^n)$.
We further need the Yamabe constant at infinity.

Definition 4. *(see [11]) Let $z \in M$ be a fixed point. We denote by $B_R \subset M$ the ball w.r.t. the metric g around z with radius R. Then,*

$$\overline{Q(M, g)} := \lim_{R\to\infty} Q(M \setminus B_R, g).$$

The limit always exists since with Theorem 3 we have $Q(M \setminus B_{R_1}, g) \leq Q(M \setminus B_{R_2}, g) \leq Q(S^n, g_{st})$ for $R_1 \leq R_2$. Hence, $\overline{Q(M)} \geq Q(M)$. Moreover, the definition is independent of the point z.

3 Solution of the Euler-Lagrange equation

The main aim of this section is to prove Theorem 1. For that, we start by considering the weighted subcritical problem. Firstly, we will prove the existence of solutions of this weighted subcritical problem, see Lemma 9. Then, the convergence of these solutions will be achieved in two steps: At first, we fix the weight ρ^α and let the subcritical exponent ($p < p_{crit}$) converge to the critical one, cf. Lemma 11. Secondly, in Lemma 12 we let $\alpha \to 0$, i.e. we establish the convergence to the unweighted critical problem.
Throughout this section let (M^n, g) be a complete connected Riemannian manifold.

We start by considering a weighted subcritical problem, see Definition 2, i.e. $2 \leq p < p_{crit}$ and $\alpha > 0$. That means we look for a solution of the Euler-Lagrange equation

$$Lv = Q_p^\alpha \rho^{\alpha p} v^{p-1} \text{ where } \|\rho^\alpha v\|_p = 1.$$

Before considering the solutions, we shortly give some preliminaries on the positivity of Q_p^α:

Lemma 5.
i) *For $0 \leq \alpha \leq \beta$ and $Q \geq 0$, we have $Q_p^\alpha \leq Q_p^\beta$ and $\lim_{\alpha\to 0} Q_p^\alpha = Q_p$.*
ii) *$Q_p^\alpha \geq \limsup_{s\to p} Q_s^\alpha$ for all $\alpha > 0$.*

Proof.
i) Since $0 < \rho \leq 1$ and $\alpha \leq \beta$, $\|\rho^\alpha v\|_p \geq \|\rho^\beta v\|_p$. With $Q \geq 0$ we know $\int_M vLv\mathrm{dvol}_g \geq 0$ for all $v \in C_c^\infty(M)$ and, hence, $Q_p^\alpha \leq Q_p^\beta$ and

$$\lim_{\alpha\searrow 0} Q_p^\alpha = \inf_{\alpha\geq 0} \inf_v \frac{\int_M vLv\,\mathrm{dvol}_g}{\|\rho^\alpha v\|_p^2} = \inf_v \inf_{\alpha\geq 0} \frac{\int_M vLv\,\mathrm{dvol}_g}{\|\rho^\alpha v\|_p^2} = \inf_v \frac{\int_M vLv\,\mathrm{dvol}_g}{\|v\|_p^2} = Q_p.$$

ii) Let now $s \leq p$. Then $\|v\|_s \to \|v\|_p$ as $s \to p$ and, thus, for all $v \in C_c^\infty(M)$ we have

$$Q_p^\alpha = \inf_v \frac{\int_M vLv\mathrm{dvol}_g}{\|\rho^\alpha v\|_p^2} = \inf_v \lim_{s\to p} \frac{\int_M vLv\mathrm{dvol}_g}{\|\rho^\alpha v\|_s^2} \geq \lim_{s\to p} \inf_v \frac{\int_M vLv\mathrm{dvol}_g}{\|\rho^\alpha v\|_s^2} = \lim_{s\to p} Q_s^\alpha. \qquad \square$$

Remark 6.
i) On closed manifolds, if $Q_p \geq 0$, there is already equality in Lemma 5.ii, cf. [15, Lem. V.2.3]. But for the Euclidean space $(\mathbb{R}^n, g_E)$ we have $Q(\mathbb{R}^n) = Q(S^n) > 0$ and $Q_s(\mathbb{R}^n) = 0$ for $s \in [2, p_{crit})$, which can be seen when rescaling a radial test function $v(r) \in C_c^\infty(\mathbb{R}^n)$ by a constant $\overline{v}(r) = v(\lambda r)$.
ii) On closed Riemannian manifolds, the signs of the Yamabe invariant Q and the first eigenvalue μ of the conformal Laplacian always coincide. On open manifolds, this is again already false for the Euclidean space where $\mu(\mathbb{R}^n) = 0$ but $Q(\mathbb{R}^n) = Q(S^n)$.

Lemma 7. *We have $\mu < 0$ if and only if $Q < 0$.*
If we assume additionally that the embedding $H_1^2 \hookrightarrow L^p$ for $2 \leq p \leq p_{crit}$ is continuous, that the scalar curvature is bounded from below and that $\mu > 0$, then $Q_p > 0$ and $\liminf_{p\to p_{crit}} Q_p > 0$.

Proof. If $\mu < 0$, there exists a function $v \in C_c^\infty(M)$ with $\int_M vLv\,\mathrm{dvol}_g < 0$. Thus, $Q_p < 0$ for all p (in particular $Q = Q_{p_{crit}} < 0$). The converse is obtained analogously.
This implies that $\mu \geq 0$ if and only if $Q_p \geq 0$. Now let there be a continuous Sobolev embedding and let $Q_p = 0$: We show by contradiction that $\mu = 0$, i.e. we argue against the assumption $\mu > 0$. Let $v_i \in C_c^\infty(M)$ be a minimizing sequence: $\|v_i\|_p = 1$ with

$\int v_i L v_i \mathrm{dvol}_g \searrow 0$. Then, since $\mu > 0$, $\|v_i\|_2 \to 0$. Hence, with the lower bound for the scalar curvature and

$$
\begin{aligned}
0 \leftarrow \int_M v_i L v_i \mathrm{dvol}_g &= a_n \|\nabla v_i\|_2^2 + \int_M \mathrm{scal}\, v_i^2 \mathrm{dvol}_g \\
&\geq a_n \|\nabla v_i\|_2^2 + \inf_M \mathrm{scal}\, \|v_i\|_2^2
\end{aligned}
$$

we get $\|\nabla v_i\|_2 \to 0$. Thus, $v_i \to 0$ in H_1^2, but the continuous Sobolev embedding gives $1 = \|v_i\|_p \leq C\|v_i\|_{H_1^2}$.
Analogously, we proceed with a minimizing sequence v_p for $\liminf_{p\to p_{crit}} Q_p = 0$, i.e. $\|v_p\|_p = 1$ and $\int_M v_p L v_p \mathrm{dvol}_g \to 0$ for $p \to p_{crit}$. This implies, exactly as before, that $\|v_p\|_{H_1^2} \to 0$. But from the embeddings, we get

$$1 = \|v\|_p \leq C(p)\|v_p\|_{H_1^2} \leq \max_{p\in[2,p_{crit}]} C(p)\|v_p\|_{H_1^2}.$$

Since each $p \in [2, p_{crit}]$ can be written as $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{p_{crit}}$ with $0 \leq \theta \leq 1$, we then get by interpolation that for all $u \in H_1^2$

$$\|u\|_p \leq \|u\|_2^{1-\theta}\|u\|_{p_{crit}}^{\theta} \leq C(2)^{1-\theta} C(p_{crit})^{\theta} \|u\|_{H_1^2}.$$

Thus, $C(p) \leq C(2)^{1-\theta} C(p_{crit})^{\theta}$ which implies that $\max_{p\in[2,p_{crit}]}$ is finite which provides a contradiction to $\liminf_{p\to p_{crit}} Q_p = 0$ (the same interpolation argument applied to $p \in [p-\epsilon, p+\epsilon]$ even shows that $C(p)$ is continuous in p). $\square$

Remark 8. For closed manifolds and $Q \geq 0$, it holds $Q(M,g) = \inf_{\overline{g}\in[g]} \mu(\overline{g})\mathrm{vol}(\overline{g})^{\frac{2}{n}}$ where $[g]$ denotes the conformal class of g. For complete manifolds and $Q \geq 0$, we have analogously that

$$Q(M,g) = \inf_{\overline{g}\in[g],\ \mathrm{vol}(\overline{g})<\infty} \mu(\overline{g})\mathrm{vol}(\overline{g})^{\frac{2}{n}}.$$

For manifolds of finite volume, this implies that from $\mu = \mu(g) = 0$ we obtain $Q = 0$.

Now, we come to solutions of the weighted subcritical problem.

Lemma 9. *Assume that the embedding $H_1^2 \hookrightarrow \rho^\alpha L^p$ is compact for all $\alpha > 0$ and $2 \leq p < p_{crit} = \frac{2n}{n-2}$. Furthermore, let $\tilde{c} \geq \mathrm{scal} \geq c$ for constants $\tilde{c}$ and c. Let $\mu > 0$.*
Then, for all $\alpha > 0$ and $2 \leq p < p_{crit}$, there exists a positive function $v \in C^\infty \cap H_1^2$ with $Lv = Q_p^\alpha \rho^{\alpha p} v^{p-1}$ and $\|\rho^\alpha v\|_p = 1$.

Proof. Firstly, from Lemma 7 we know that $Q > 0$ and, thus, $Q_p^\alpha > 0$ for all $\alpha > 0$. Let now $\alpha > 0$ and $2 \leq p < p_{crit}$ be fixed. Moreover, let $v_i \in C_c^\infty(M)$ be a minimizing sequence for Q_p^α, i.e. $\int_M v_i L v_i \,\mathrm{dvol}_g \searrow Q_p^\alpha$ and $\|\rho^\alpha v_i\|_p = 1$. Without loss of generality, we can assume that v_i is nonnegative. Moreover, with

$$0 \leq Q_p^\alpha \swarrow \int_M v_i L v_i \mathrm{dvol}_g = a_n \|\nabla v_i\|_2^2 + \int_M \mathrm{scal}\ \ v_i^2 \,\mathrm{dvol}_g \geq \mu \|v_i\|_2^2$$

for $i \to \infty$ and $\mu > 0$, we obtain that $\|v_i\|_2$ is uniformly bounded. Hence, using that $\int_M v_i L v_i \mathrm{dvol}_g \geq a_n \|dv_i\|_2^2 + c\|v_i\|_2^2$ the sequence v_i is uniformly bounded in H_1^2. So, $v_i \to v \geq 0$ weakly in H_1^2 with $\|\nabla v\|_2 \leq \liminf \|\nabla v_i\|_2$. Due to the compactness of the Sobolev embedding, $\rho^\beta v_i$ converges to $\rho^\beta v$ even strongly both in L^p and in L^2 for all $\beta > 0$. Hence, for $\beta = \alpha$ we obtain $\|\rho^\alpha v\|_p = 1$. Moreover, for any $w \in L^2$ $\rho^\beta w \nearrow w$ pointwise and, thus,

with the Lemma of Fatou we get $\|w\|_2^2 = \|\liminf_{\beta\to 0}\rho^\beta w\|_2^2 \le \liminf_{\beta\to 0}\|\rho^\beta w\|_2^2 \le \|v\|_2^2$. Since the scalar curvature is bounded, we further get $\int_M \mathrm{scal}\,\rho^{2\beta} v_i^2\,\mathrm{dvol}_g \to \int_M \mathrm{scal}\, v_i^2\,\mathrm{dvol}_g$ for $\beta \to 0$ and, hence, for every $\epsilon > 0$ we have for i large enough that

$$\int_M \mathrm{scal}\, v^2 \mathrm{dvol}_g \xleftarrow{\beta\to 0} \int_M \mathrm{scal}\,\rho^{2\beta} v^2 \le \int_M \mathrm{scal}\,\rho^{2\beta} v_i^2\,\mathrm{dvol}_g + \epsilon \xrightarrow{\beta\to 0} \int_M \mathrm{scal}\, v_i^2 \mathrm{dvol}_g + \epsilon.$$

Thus,

$$\begin{aligned}\int_M vLv\,\mathrm{dvol}_g = a_n\|\nabla v\|_2^2 + \int_M \mathrm{scal}\, v^2\,\mathrm{dvol}_g &\le a_n \liminf_{i\to\infty}\|\nabla v_i\|_2^2 + \lim_{i\to\infty}\int_M \mathrm{scal}\, v_i^2\,\mathrm{dvol}_g \\ &\le \lim_{i\to\infty}\int_M v_i L v_i\,\mathrm{dvol}_g = Q_p^\alpha.\end{aligned}$$

Hence, $\|\rho^\alpha v\|_p^2 \int_M vLv\mathrm{dvol}_g \le Q_p^\alpha$. But since Q_p^α is the minimum, it already holds equality and v fulfills the Euler-Lagrange equation $Lv = Q_p^\alpha \rho^{\alpha p} v^{p-1}$ with $\|\rho^\alpha v\|_p = 1$. Furthermore, there is a constant $C > 0$ with $\Delta v + Cv \ge 0$. Thus, due to the maximum principle, v is everywhere positive. From local elliptic regularity theory, we know that v is smooth. □

Before considering the convergence of solutions, we observe that

Remark 10. From Lemma 5.i it follows:
Let $Q(\mathbb{R}^n, g_E) > Q(M)$. Then, there exists an $\alpha_0 > 0$ such that for all $0 \le \alpha \le \alpha_0$ we have $Q(\mathbb{R}^n) > Q_{p_{crit}}^\alpha(M)$.

Next, we show that a suitable subsequence of the weighted subcritical solutions as in Lemma 9 converges to a solution of the weighted critical problem, i.e. we fix α and let $p \to p_{crit}$:

Lemma 11. *Let $v_{\alpha,p} \in H_1^2$ ($\alpha > 0$, $p < p_{crit}$) be smooth positive solutions of $Lv_{\alpha,p} = Q_p^\alpha \rho^{\alpha p} v_{\alpha,p}^{p-1}$ with $\|\rho^\alpha v_{\alpha,p}\|_p = 1$. We assume that $Q(\mathbb{R}^n, g_E) > Q(M)$. Furthermore, let M have bounded geometry and $\mu > 0$. Let $\alpha < \alpha_0$ be fixed where α_0 is chosen as in Remark 10. Then, abbreviating $v_p = v_{\alpha,p}$*

a) there exists $k > 0$ such that $\sup v_p \le k$ for all p

b) for $p \to p_{crit}$, $v_p \to v_\alpha \ge 0$ in C^2-topology on each compact set and

$$Lv_\alpha = Q_{p_{crit}}^\alpha \rho^{\alpha p_{crit}} v_\alpha^{p_{crit}-1} \text{ with } \|\rho^\alpha v_\alpha\|_{p_{crit}} = 1.$$

Proof. From Lemma 22 in the Appendix we know that v_p has a maximum.
a) Let $x_p \in M$ be a point where v_p attains its maximum. We prove the claim by contradiction and assume that $m_p := v_p(x_p) \to \infty$.
If, for $p \to p_{crit}$, the sequence x_p converges to a point $x \in M$, we could simply use Schoen's argument [15, pp. 204-206] and introduce geodesic normal coordinates around x to show that m_p is bounded from above by a constant independent of p.
In general, the sequence x_p can escape to infinity, that is why we take a normal coordinate system around each x_p with radius $\epsilon := inj(M)$. This coordinate system will be denoted by ϕ_p and $\phi_p : B_\epsilon(0) \subset \mathbb{R}^n \to M$ with $\phi_p(0) = x_p$. The bounded geometry of M and the boundedness of each v_p ensures that Schoen's argument can be adapted:
With respect to the geodesics coordinates introduced above, we have the following expansions [12, pp. 60-61]

$$\begin{aligned} g_{rq}^p(x) &= \delta_{rq} + \frac{1}{3} R_{rijq}^p x^i x^j + O(|x^3|) \\ \det g_{rq}^p(x) &= 1 - \frac{1}{3} R_{ij}^p x^i x^j + O(|x^3|), \end{aligned}$$

where the upper index p always refers to the coordinate system ϕ_p around x_p, R^p_{pijq} denotes the Riemannian curvature in x_p and R^p_{ij} the corresponding Ricci curvature. After rescaling $u_p = {m_p}^{-1} v_p(\phi_p(\delta_p x))$ with $\delta_p = m_p^{(2-p)/2} \to 0$ (note that $\delta_p \to 0$ as $p \to p_{crit}$) we have $u_p : B_{\frac{\epsilon}{\delta_p}}(0) \to M$ with $u_p(0) = 1$, $u_p \leq 1$. The weight function in the new coordinates will be denoted by $\rho_p(x) := \rho(\phi_p(\delta_p x))$.
In the following, we identify $\phi_p(\delta_p x)$ with $\delta_p x$ and omit ϕ_p in the notation.
The Euler-Lagrange equation in the geodesic coordinates reads (compare [15])

$$\frac{1}{b_p}\partial_j(b_p a_p^{ij}\partial_i u_p) - c_p u_p + Q_p^\alpha \rho_p^{\alpha p} u_p^{p-1} = 0 \tag{2}$$

where

$$\begin{aligned} a_p^{ij}(x) &= a_n g^{ij}(\delta_p x) \to a_n \\ b_p(x) &= \sqrt{\det\, g(\delta_p x)} \to 1 \\ c_p(x) &= m_p^{1-p}\mathrm{scal}\,(\delta_p x) \to 0 \end{aligned} \tag{3}$$

for $p \to p_{crit}$.
Since the Riemannian curvature and, hence, the scalar curvature are bounded, the convergences in (3) is C^1 on any compact subset of $\mathbb{R}^n$.
Now, we can follow the proof of Schoen and we will show with interior Schauder and global L^p estimates that u_p is bounded in $C^{2,\gamma}$ (for appropriate γ) on each compact subset K and, thus, obtain $u_p \to u$ on C^2 in K: We have on a compact subset $K \subset \Omega \subset \mathbb{R}^n$:
L^p estimate (using $\rho_p \leq 1$ and $u_p \leq 1$):

$$\|u_p\|_{H_2^p(K)} \leq C_K(\|u_p\|_{L^q(K)} + \|u_p^{p-1}\|_{L^q(K)}) \leq 2C_K \mathrm{vol}(K)^{\frac{1}{q}} \leq C(K),$$

where q and p are conjugate and $C(K)$ only depends on the subset K.
Together with the continuous embedding $H_1^q \hookrightarrow C^{0,\gamma}$ where $\gamma \leq 1 - \frac{n}{q}$, we obtain, that u_p and, thus, also u_p^{p-1}, are uniformly bounded in $C^{0,\gamma}(K)$ (for possibly smaller γ). With the interior Schauder estimate

$$\|u_p\|_{C^{2,\gamma}(K)} \leq C(\|u_p\|_{C^0(\Omega)} + \|u_p^{p-1}\|_{C^{0,\gamma}(\Omega)})$$

u_p is uniformly bounded in $C^{2,\gamma}(K)$. With the theorem of Arcela-Ascoli, we obtain, by going to a subsequence if necessary, that $u_p \to u$ in C^2 on each compact subset. Thus, $u \geq 0$ and $u(0) = 1$.
We estimate

$$\int_{|x|<\epsilon\delta_p^{-1}} u_p^p b_p \,\mathrm{dvol}_{g_E} = \int_{B_\epsilon(x_p)} \delta_p^{\frac{2p}{p-2}-n} v_p^p \,\mathrm{dvol}_g \leq C\delta_p^{\frac{2p}{p-2}-n}\|v_p\|^p_{H_1^2(M)}$$

where the equality is obtained by change of variables and the inequality is the Sobolev embedding (see Theorem 19). Using $Lv_p = Q_p^\alpha \rho^{\alpha p} v_p^{p-1}$ with $\|\rho^\alpha v_p\|_p = 1$, we obtain

$$\begin{aligned} Q_p^\alpha = \int_M v_p L v_p \mathrm{dvol}_g &= a_n\|\nabla v_p\|^2_{L^2(M)} + \int_M \mathrm{scal}\, v_p^2 \mathrm{dvol}_g \\ &\geq a_n \|\nabla v_p\|^2_{L^2(M)} + \inf \mathrm{scal}\, \|v_p\|^2_{L^2(M)} \end{aligned}$$

and, thus,

$$\int_{|x|<\epsilon\delta_p^{-1}} u_p^p b_p \,\mathrm{dvol}_{g_E} \leq C\delta_p^{\frac{2p}{p-2}-n}\left(\|v_p\|_{L^2(M)} + \left(a_n^{-1}\left(Q_p^\alpha - \inf \mathrm{scal}\,\|v_p\|^2_{L^2}\right)\right)^{\frac{1}{2}}\right)^p.$$

From $\mu > 0$, we have additionally that $\|v_p\|_{L^2}^2 \leq \mu^{-1} \int v_p L v_p \mathrm{dvol}_g = \mu^{-1} Q_p^\alpha$. With $\limsup_{p\to p_{crit}} Q_p^\alpha \leq Q_{p_{crit}}^\alpha$ (Lemma 5.ii), we get that $\|v_p\|_{L^2}$ is uniformly bounded on $p \in (2, p_{crit})$. Moreover, $\frac{2p}{p-2} - n \searrow 0$ for $p \to p_{crit}$. Hence, the integral $\int_{|x|<\epsilon\delta_p^{-1}} u_p^p b_p \,\mathrm{dvol}_{g_E}$ is bounded from above by a constant independent of p. Thus, by the Lemma of Fatou $u \in L^{p_{crit}}(\mathbb{R}^n)$.

In order to construct a contradiction we distinguish between two cases:
At first, we consider the case that x_p escapes to infinity if $p \to p_{crit}$:
Then, $\rho_p \to 0$ as $p \to p_{crit}$ and

$$a_n \Delta u = \limsup_{p\to p_{crit}} (Q_p^\alpha(M) \rho_p^{\alpha p} u_p^{p-1}) = 0$$

on $\mathbb{R}^n$. From the maximum principle, $u(0) = 1$ and $u \leq 1$ we obtain that $u \equiv 1$ which contradicts $u \in L^{p_{crit}}(\mathbb{R}^n)$.

Secondly, we consider the remaining case that a subsequence of x_p converges to a point $y \in M$. Then ρ_p converges to the constant $\rho(y)$. Hence,

$$\begin{aligned} a_n \Delta u &= \limsup_{p\to p_{crit}} (Q_p^\alpha(M) \rho_p^{\alpha p} u_p^{p-1}) \\ &= (\limsup_{p\to p_{crit}} Q_p^\alpha(M)) \rho^{\alpha p_{crit}}(y) u^{p_{crit}-1} \leq Q_{p_{crit}}^\alpha(M) \rho^{\alpha p_{crit}}(y) u^{p_{crit}-1} \end{aligned}$$

on $\mathbb{R}^n$. With $u \geq 0$ and $u(0) = 1$, we obtain $u > 0$ from the maximum principle.
Moreover, we have for $\epsilon_1 \leq \epsilon$

$$\begin{aligned} \int_{|x|<\epsilon_1\delta_p^{-1}} u_p^p b_p \,\mathrm{dvol}_{g_E} &\leq (\min_{B_{\epsilon_1}(x_p)} \rho^{\alpha p})^{-1} \int_{B_{\epsilon_1}(x_p)} \rho^{\alpha p} v_p^p \delta_p^{\frac{2p}{p-2}-n} \,\mathrm{dvol}_g \\ &\leq (\min_{B_{\epsilon_1}(x_p)} \rho^{\alpha p})^{-1} \delta_p^{\frac{2p}{p-2}-n} \leq (\min_{B_{\epsilon_1}(x_p)} \rho^{\alpha p})^{-1} \to \max_{B_{\epsilon_1}(y)} \rho^{-\alpha p_{crit}} \end{aligned}$$

for $p < p_{crit}$ and by Fatou's Lemma, we obtain $\|u\|_{p_{crit}, g_E} \leq \max_{B_{\epsilon_1}(y)} \rho^{-\alpha}$. Letting $\epsilon_1 \to 0$ we have $\|u\|_{p_{crit}, g_E} \leq \rho^{-\alpha}(y)$. Thus,

$$\begin{aligned} Q(\mathbb{R}^n) &\leq \frac{\int a_n u \Delta u \,\mathrm{dvol}_{g_E}}{\|u\|_{p_{crit}, g_E}^2} \leq Q_{p_{crit}}^\alpha(M) \rho^{\alpha p_{crit}}(y) \|u\|_{p_{crit}, g_E}^{\frac{p_{crit}-2}{p_{crit}}} \\ &\leq Q_{p_{crit}}^\alpha(M) \rho^{\alpha p_{crit}}(y) \rho^{-\alpha(p_{crit}-2)}(y) \leq Q_{p_{crit}}^\alpha(M) \rho^{2\alpha}(y) \\ &\leq Q_{p_{crit}}^\alpha(M), \end{aligned}$$

which contradicts the assumption that $Q(\mathbb{R}^n, g_E) > Q(M)$ and $\alpha \leq \alpha_0$ (see Remark 10). Thus, there exists an $k > 0$ with $m_p \leq k$.
b) From a), we know $\max v_p \leq k$ for all p. Thus, we can apply the interior Schauder and global L^p-estimates as above and obtain, that $v_p \to v_\alpha$ in C^2 on each compact subset K. Together with Lemma 5, we get

$$L v_\alpha = (\limsup_{p\to p_{crit}} Q_p^\alpha) \rho^{\alpha p_{crit}} v_\alpha^{p_{crit}-1} \leq Q_{p_{crit}}^\alpha \rho^{\alpha p_{crit}} v_\alpha^{p_{crit}-1}.$$

Clearly, $\|\rho^\alpha v_\alpha\|_{p_{crit}} \leq 1$ and smoothness of v_α follows from standard elliptic regularity theory.

It remains to show that $\|\rho^\alpha v_\alpha\|_{p_{crit}} = 1$. Firstly, we assume that $v_\alpha = 0$:
Since

$$Q_p \le \frac{\int_M v_p L v_p \mathrm{dvol}_g}{\left(\int_M v_p^p \mathrm{dvol}_g\right)^{\frac{2}{p}}} = Q_p^\alpha \|v_p\|_p^{-2}$$

and $Q_p > 0$ (Lemma 7), we have

$$\limsup_{p\to p_{crit}} \|v_p\|_p \le \limsup_{p\to p_{crit}} \left(\frac{Q_p^\alpha}{Q_p}\right)^{\frac{1}{2}} \le \left(\frac{Q_{p_{crit}}^\alpha}{\liminf_{p\to p_{crit}} Q_p}\right)^{\frac{1}{2}} =: c < \infty$$

where the last inequality uses $\liminf_{p\to p_{crit}} Q_p > 0$ from Lemma 7. Thus,

$$\limsup_{p\to p_{crit}} \int_{M\setminus B_R} \rho^{\alpha p} v_p^p \mathrm{dvol}_g \le \limsup_{p\to p_{crit}} (\max_{M\setminus B_R} \rho^{\alpha p}\ \|v_p\|_p^p) \le e^{-(R-\xi)\alpha p_{crit}} c^{\frac{p_{crit}}{2}}$$

where the last inequality follows with Remark 18.
Choose $R = R(\alpha)$ big enough such that $\limsup_{p\to p_{crit}} \int_{M\setminus B_R} \rho^{\alpha p} v_p^p \mathrm{dvol}_g \le \frac{1}{2}$. Then,

$$\limsup_{p\to p_{crit}} \int_{B_R} \rho^{\alpha p} v_p^p \mathrm{dvol}_g \ge \frac{1}{2},$$

which contradicts the assumption that $v_p \to v_\alpha = 0$. Thus, $\|\rho^\alpha v_\alpha\|_{p_{crit}} > 0$.
Using the smoothness of v_α and that it weakly fulfills $L v_\alpha \le Q_{p_{crit}}^\alpha \rho^{\alpha p_{crit}} v_\alpha^{p_{crit}-1}$, we can compute

$$0 < Q_{p_{crit}}^\alpha \le \frac{\int_M v_\alpha L v_\alpha \mathrm{dvol}_g}{\left(\int_M \rho^{\alpha p_{crit}} v_\alpha^{p_{crit}} \mathrm{dvol}_g\right)^{\frac{2}{p_{crit}}}} \le Q_{p_{crit}}^\alpha \|\rho^\alpha v_\alpha\|_{p_{crit}}^{p_{crit}-2}$$

and obtain $\|\rho^\alpha v_\alpha\|_{p_{crit}} = 1$ and, hence, equality in $L v_\alpha = Q_{p_{crit}}^\alpha \rho^{\alpha p_{crit}} v_\alpha^{p_{crit}-1}$.
In particular, we have $\limsup_{p\to p_{crit}} Q_p^\alpha = Q_{p_{crit}}^\alpha$. ☐

Similarly, we now take the limit for $\alpha \to 0$:

Lemma 12. *Let $v_\alpha \in H_1^2$ ($\alpha_0 \ge \alpha > 0$) be smooth and positive solutions of $L v_\alpha = Q_{p_{crit}}^\alpha \rho^{\alpha p_{crit}} v_\alpha^{p_{crit}-1}$ with $\|\rho^\alpha v_\alpha\|_{p_{crit}} = 1$. Furthermore, let M have bounded geometry and let $Q(\mathbb{R}^n, g_E) > Q(M)$.*
Then, there exists $k > 0$ such that $\sup v_\alpha \le k$ for all α and for $\alpha \to 0$, $v_\alpha \to v$ in C^2-topology on each compact set and $L v = Q_{p_{crit}} v^{p_{crit}-1}$.
If additionally $\overline{Q(M,g)} > Q(M,g)$, we have $\|v\|_{p_{crit}} = 1$.

Proof. The first part is proven in the same way as in Lemma 11: Let $x_\alpha \in M$ be points where v_α attains its maximum $m_\alpha := v_\alpha(x_\alpha)$. We assume that $m_\alpha \to \infty$. In the same way as in Lemma 11 we introduce rescaled geodesic coordinates ϕ_α on $B_\epsilon(x_\alpha)$ (where ϵ is still the injectivity radius of M) and obtain $u_\alpha = m_\alpha^{-1} v_\alpha(\phi_\alpha(\delta_\alpha x))$ with $\delta_\alpha = m_\alpha^{(2-p_{crit})/2}$ that fulfills the same (after changing the upper index p to p_{crit} and the lower p to α) Euler-Lagrange equation (2). Using interior Schauder and global L^p-estimates, one can again prove that $u_\alpha \in H_1^{q_{crit}}$ and, thus, uniformly bounded in $C^{0,\gamma}(K)$ for compact subsets $K \subset M$ and appropriate γ. Hence, $u_\alpha \to u$ in C^2 on compact subsets with $u \ge 0$ and $u(0) = 1$.
An analogous estimate as in Lemma 11 shows that $\int_{|x|<\epsilon\delta_\alpha^{-1}} u_\alpha^{p_{crit}} b_\alpha \mathrm{dvol}_{g_E}$ is bounded (independent on α). Thus, the lemma of Fatou gives $u \in L^{p_{crit}}(\mathbb{R}^n)$ and

$$a_n u \Delta u = \limsup_{\alpha\to 0} (Q_{p_{crit}}^\alpha \rho_\alpha^{\alpha p_{crit}} u_\alpha^{p_{crit}-1}) \le Q u^{p_{crit}-1} \limsup_{\alpha\to 0} \max_{B_\epsilon(x_\alpha)} \rho^{\alpha p_{crit}}.$$

With Remark 18 we get

$$\begin{aligned}a_n u\Delta u &\le Qu^{p_{crit}-1}\limsup_{\alpha\to 0}\max_{B_\epsilon(x_\alpha)} e^{-\alpha p_{crit}(|x|-\xi)} \le Qu^{p_{crit}-1}\limsup_{\alpha\to 0} e^{-\alpha p_{crit}(|x_\alpha|-\xi-\epsilon)}\\ &= Qu^{p_{crit}-1}\limsup_{\alpha\to 0} e^{-\alpha p_{crit}|x_\alpha|}.\end{aligned}$$

In case that $\alpha|x_\alpha|\to\infty$ as $\alpha\to 0$, the last limes goes to zero and this leads to a contradiction as in Lemma 11 where the case $|x_p|\to\infty$ as $p\to p_{crit}$ was discussed. Thus, from now on we can assume that $\alpha|x_\alpha|$ is bounded.
Moreover, we can estimate as in Lemma 11 that

$$\int_{|x|<\epsilon\delta_\alpha^{-1}} u_\alpha^{p_{crit}} b_\alpha \mathrm{dvol}_{g_E} \le \max_{B_\epsilon(x_\alpha)} \rho^{-\alpha p_{crit}}.$$

and with Remark 18 we get

$$\int_{|x|<\epsilon\delta_\alpha^{-1}} u_\alpha^{p_{crit}} b_\alpha \mathrm{dvol}_{g_E} \le \max_{B_\epsilon(x_\alpha)} e^{\alpha p_{crit}(|x|+\xi)} = e^{\alpha p_{crit}(|x_\alpha|+\epsilon+\xi)}$$

and, hence, $\|u\|_{p_{crit},g_E}\le \liminf_{\alpha\to 0} e^{\alpha(|x_\alpha|+\epsilon+\xi)} = \liminf_{\alpha\to 0} e^{\alpha|x_\alpha|}$.
Thus,

$$\begin{aligned}Q(\mathbb{R}^n) &\le \frac{\int a_n u\Delta u \,\mathrm{dvol}_{g_E}}{\|u\|^2_{p_{crit},g_E}} \le Q(M)\liminf_{\alpha\to 0} e^{\alpha|x_\alpha|}\|u\|_{p_{crit},g_E}^{\frac{p_{crit}-2}{p_{crit}}}\\ &\le Q(M)\liminf_{\alpha\to 0} e^{\alpha|x_\alpha|}\limsup_{\alpha\to 0} e^{-\alpha|x_\alpha|} = Q(M)\end{aligned}$$

where the last equality follows since both limits in $\liminf_{\alpha\to 0} e^{\alpha|x_\alpha|}\limsup_{\alpha\to 0} e^{-\alpha|x_\alpha|}$ are finite since we assumed that $\alpha|x_\alpha|$ is bounded. But this gives a contradiction to $Q(\mathbb{R}^n) > Q(M)$. Hence, v_α has to be bounded uniformly in α.
Then we can again use interior Schauder and global L^p estimates and obtain $v_\alpha\to v$ in C^2 on compact subsets with $Lv = Qv^{p_{crit}-1}$.
Assume now that $\overline{Q(M,g)} > Q(M,g)\ge 0$. Clearly, also $\rho^\alpha v_\alpha\to v$ in C^2 on compact subsets, $\|v\|_{p_{crit}}\le 1$ and smoothness of v follows again from elliptic regularity theory. We have to show that $\|v\|_{p_{crit}} = 1$.
Firstly assume, that $v_\alpha\to v\equiv 0$. Then, for a fixed ball $B_r(z)$ around $z\in M$ with radius r we get that

$$\begin{aligned}Q(M) &= \liminf_{\alpha\to 0} Q^\alpha_{p_{crit}}(M) = \liminf_{\alpha\to 0}\int_M v_\alpha L v_\alpha \,\mathrm{dvol}_g\\ &\ge \liminf_{\alpha\to 0}\int_{M\setminus B_r} v_\alpha L v_\alpha \,\mathrm{dvol}_g + \liminf_{\alpha\to 0}\int_{B_r} v_\alpha L v_\alpha \,\mathrm{dvol}_g,\end{aligned}$$

where the first equality is given by Lemma 5.i and the second equality follows from $Lv_\alpha = Q^\alpha_{p_{crit}} v^{p_{crit}-1}$ and $\|\rho^\alpha v_\alpha\|_{p_{crit}} = 1$. The last summand vanishes as $\alpha\to 0$. In order to estimate the other summand, we introduce a smooth cut-off function $\eta_r\le 1$ with support in $M\setminus B_r$ and $\eta_r\equiv 1$ on $M\setminus B_{2r}$. Then, for $\alpha\to 0$

$$\begin{aligned}&\left|\int_{M\setminus B_r} \eta_r v_\alpha L(\eta_r v_\alpha)\,\mathrm{dvol}_g - \int_{M\setminus B_r} v_\alpha L v_\alpha \,\mathrm{dvol}_g\right|\\ &= \left|\int_{B_{2r}\setminus B_r} \eta_r v_\alpha L(\eta_r v_\alpha)\,\mathrm{dvol}_g - \int_{B_{2r}\setminus B_r} v_\alpha L v_\alpha \,\mathrm{dvol}_g\right| \to 0.\end{aligned}$$

since $v_\alpha \to 0$ in C^2 on each compact set. Hence, with $\int_M v_\alpha^{p_{crit}} \mathrm{dvol}_g \geq \int_M (\rho^\alpha v_\alpha)^{p_{crit}} \mathrm{dvol}_g = 1$ and Lemma 5.i we obtain

$$
\begin{aligned}
Q(M) &= \liminf_{\alpha\to 0} Q^\alpha_{p_{crit}}(M) \geq \liminf_{\alpha \to 0} \int_{M\setminus B_r} \eta_r v_\alpha L(\eta_r v_\alpha)\, \mathrm{dvol}_g \\
&\geq \liminf_{\alpha\to 0} Q^\alpha_{p_{crit}}(M\setminus B_r) \left(\int_{M\setminus B_r} (\eta_r v_\alpha)^{p_{crit}}\, \mathrm{dvol}_g \right)^{\frac{2}{p_{crit}}} \\
&= \liminf_{\alpha\to 0} Q^\alpha_{p_{crit}}(M\setminus B_r) \left(\int_M v_\alpha^{p_{crit}} \mathrm{dvol}_g - \int_{B_{2r}\setminus B_r} (1-\eta_r^{p_{crit}}) v_\alpha^{p_{crit}}\, \mathrm{dvol}_g \right)^{\frac{2}{p_{crit}}} \\
&\geq Q(M\setminus B_r),
\end{aligned}
$$

where the integral over $B_{2r}\setminus B_r$ again vanishes since $v_\alpha \to 0$ on compact sets.
Thus, $\overline{Q(M)} \leq Q(M)$ which contradicts the assumption. Thus, we have $\|v\|_{p_{crit}} > 0$.
Since $Lv = Qv^{p_{crit}-1}$ and $v \in H_1^2$, we further obtain that

$$
Q \leq \frac{\int_M vLv\, \mathrm{dvol}_g}{\|v\|^2_{p_{crit}}} = Q\|v\|^{p_{crit}-2}_{p_{crit}} \leq Q,
$$

i.e. $\|v\|_{p_{crit}} = 1$. □

Proof of Theorem 1. Combining Lemma 9 and 12 with [17, Cor. 2] (cf. Appendix A) where the required Sobolev embeddings are proven for manifolds of bounded geometry, we obtain Theorem 1. □

For homogeneous manifolds with strictly positive scalar curvature, we can drop the assumption on the Yamabe invariant at infinity and reprove a result of Akutagawa:

Theorem 13. *Let (M, g) be a manifold of bounded geometry,* scal $\geq c > 0$ *for a constant c and $Q(S^n) > Q(M)$. Furthermore, we assume that there exists a relatively compact set $U \subset\subset M$ such that for all $x \in M$ there is an isometry $f: M \to M$ with $f(x) \in U$. Then, there is a positive smooth solution $v \in H_1^2$ of the Euler-Lagrange equation $Lv = Q(M)v^{p_{crit}-1}$ with $\|v\|_{p_{crit}} = 1$.*

Proof. Due to the existence of the isometries, M has bounded geometry. Moreover, since the scalar curvature is uniformly positive, μ and Q are positive. Hence, with Lemma 9, we obtain positive solutions $v_{\alpha,p} > 0$ ($\alpha > 0$, $p \in [2, p_{crit})$) of $Lv_{\alpha,p} = Q_p^\alpha \rho^{\alpha p} v_{\alpha,p}^{p-1}$ with $\|\rho^\alpha v_{\alpha,p}\|_p = 1$. Lemma 11 and 12 show that for a certain subsequence $v_p = v_{\alpha(p),p}$ converges to v in C^2-topology on each compact set and $Lv \leq Qv^{p-1}$. We need to show that $\|v\|_{p_{crit}} = 1$: Due to Lemma 22, each v_p has a maximum. With the isometries, we can always pull the point x_p where v_p attains its maximum into the subset U.
Thus, w.l.og. we assume that $x_p \in U$. Since v_p is maximal in x_p, we have that $\Delta v_p(x_p) \geq 0$ and, thus, $Qv_p^{p-2}(x_p) \geq \mathrm{scal}\,(x_p) \geq c$. Let $x \in \overline{U}$ be the limit of a convergent subsequence of x_p. Then $Qv^{p_{crit}-2}(x) \geq c > 0$. Since $Q > 0$, we have $0 < \|v\|_{p_{crit}}$ and, thus, as in the proof of Lemma 12, $\|v\|_{p_{crit}} = 1$. Hence, we have a positive solution $v \in H_1^2$ of $Lv = Qv^{p-1}$ with $\|v\|_{p_{crit}} = 1$. □

Remark 14. If there exist such isometries, as described in Theorem 13, we have $\overline{Q(M)} = Q(M)$.
This can be seen when taking a minimizing sequence $v_i \in C_c^\infty(M)$ with $\|v_i\|_{p_{crit}} = 1$ and $\int_M v_i L v_i \mathrm{dvol}_g \to Q(M)$. Denote the diameter of $\operatorname{supp} v_i \cup U$ by d_i. Let $y \in M$ be

fixed. We define $\tilde{v_i} = v_i \circ f_i$ where f_i is an isometry that maps a given point $x \in M$ with $\mathrm{dist}(x, U) = i + d_i$ to a point in U Then, $\tilde{v_i} \in C_c^\infty(M \setminus B_i(y))$, $\int_M \tilde{v_i} L \tilde{v_i} \mathrm{dvol}_g \to Q(M)$ and $\|\tilde{v_i}\|_{p_{crit}} = 1$. Thus, $\overline{Q(M)} = Q(M)$.

Example 15. Consider the model spaces $(Z = S^{n-k-1} \times \mathbb{H}^{k+1}, g_c = e^{-2ct} g_{S^{n-k-1}} + g_{\mathbb{H}^{k+1}})$ that appear in [3] formed as the warped product of the standard sphere and the standard hyperbolic space for a constant $c \in [-1, 1]$. Those spaces have the required symmetries. Their scalar curvature is constant and given by $\mathrm{scal}_{g_c} = -k(k+1)c^2 + (n-k-1)(n-k-2)$, e.g. for $k < \frac{n-2}{2}$ the scalar curvature is positive for all $c \in [-1, 1]$. Note that for $c = 1$ (Z, g_1) is conformal to $S^n \setminus S^k$ and thus $Q(Z, g_1) = Q(S^n)$.
Assuming that c is chosen such that scal_{g_c} is positive and $Q(Z, g_c) < Q(S^n)$, Theorem 13 shows that for those spaces there is a solution of the Euler-Lagrange equation.

Moreover, in [3], besides the Yamabe invariant from above the following invariant is used:

$$\mu^{(1)}(M, g) = \inf\{\mu \in \mathbb{R} \mid \exists u \in L^\infty \cap L^2, u \neq 0, \|u\|_{p_{crit}} \leq 1 : \ L_g u = \mu u^{p_{crit}-1}\}.$$

The proof of [3, Lem. 3.5] shows, that if (M, g) is a complete Riemannian manifold it is $\mu^{(1)}(M, g) \geq Q(M, g)$.

Corollary 16. *Let the assumptions of Theorem 1 or of Theorem 13 be fulfilled for a manifold (M, g). Then $\mu^{(1)}(M, g) = Q(M, g)$.*

Proof. From Theorem 1 or 13 we know that there is a smooth solution $v \in H_1^2$ with $L_g v = Q v^{p_{crit}-1}$ and $\|v\|_{p_{crit}} = 1$. That solution is obtained as limit of smooth subcritical solutions $v_p \in H_1^2$. Due to Lemma 11, $v_p \leq k$ for a fixed constant k. Thus, $v \in L^\infty$ and, hence, with $\mu^{(1)} \geq Q$ from above $Q(M, g) = \mu^{(1)}(M, g)$. ☐

A Embeddings on manifolds of bounded geometry

In [8, Cor. 3.19] there are already given continuous Sobolev embeddings for manifolds of bounded geometry:

Theorem 17. *[8, Thm. 3.18 and Cor. 3.19] Let (M^n, g) be a manifold of bounded geometry. Then $H_1^q(M)$ is continuously embedded in $L^p(M)$ for $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$.*

But unfortunately those embeddings are not compact. Therefore, we will work with weighted Sobolev embeddings:
Let $\rho : M \to (0, \infty)$ be a radial admissible weight, see [17, Def. 2 and 4].

Remark 18. In the following, we will choose $\rho(x) = \exp(-r)$ where r is a smooth function with $|r(x) - |x|| < \xi$ for all $x \in M$ and a fixed $\xi > 0$ where $|x| := \mathrm{dist}(x, z)$ for fixed $z \in M$. On manifolds of bounded geometry, such a function r always exists [16, Lem. 2.1.].

We define the weighted L^p-space $\rho^\alpha L^p := \{f \mid \rho^\alpha f \in L^p(M)\}$ equipped with the norm $\|f\|_{\rho^\alpha L^p} := \|\rho^\alpha f\|_{L^p}$.

Theorem 19. *[17, Cor. 2] If the manifold (M, g) has bounded geometry, for each $2 \leq p < p_{crit} = \frac{2n}{n-2}$ the Sobolev embedding $H_1^2 \hookrightarrow \rho^\alpha L^p$ is continuous for $\alpha \geq 0$ and compact for $\alpha > 0$.*

The hard part of the above theorem is to establish compactness. The continuous embeddings can be carried over from the local case as seen below:

Remark 20. Let (M, g) be a manifold with bounded geometry.
i) (Inner L^p-estimate)[6, proof of Thm. 8.8] For all $\epsilon > 0$ there exists a constant $C_\epsilon(q)$ such that for all $x \in M$

$$\|u\|_{H_2^q(B_\epsilon(x))} \le C_\epsilon(q)(\|u\|_{L^q(B_\epsilon(x))} + \|f\|_{L^q(B_\epsilon(x))})$$

for all q, $f \in L^q_{loc}$ and where $u \in H^q_{2,loc}$ is a solution of $Lu = f$.
ii) (Imbedding) Let $n < q$ and $0 \le \gamma \le 1 - \frac{n}{q}$. From the proof of [6, Sect. 7.8 (Thm 7.26)] we have that for all $\epsilon > 0$ there exists a constant C such that for all $x \in M$ $H_2^q(B_\epsilon(x))$ is continuously embedded in $C^{0,\gamma}(\overline{B_\epsilon(x)})$

Corollary 21. *The inner L^p-estimates and the imbedding of Remark 20 hold globally on manifold of bounded geometry, i.e. i) There is a constant $C > 0$ such that for $u \in H_2^q$ and $f \in L^q$ with $Lu = f$ it holds*

$$\|u\|_{H_2^q} \le C(\|u\|_{L^q} + \|f\|_{L^q}).$$

ii) Let $n < q$ and $0 \le \gamma \le 1 - \frac{n}{q}$. There exists a constant C such that H_2^q is continuously embedded in $C^{0,\gamma}$.

Proof. We choose a countable covering of M by geodesic balls B_i all of radius $\epsilon < \mathrm{inj}(M)$. Moreover, the covering is chosen such that it is of (finite) multiplicity m, i.e. the maximal number of subsets with nonempty intersection is m, cf. [17, Sect. 2.1]. Let χ_i be a subordinated partition of unity.
Let $u \in H_2^q$ and $f \in L^q$ with $Lu = f$. Then,

$$\begin{aligned}\|u\|_{H_2^q(M)} &\le \sum_i \|\chi_i u\|_{H_2^q(B_i)} \le C(\epsilon) \sum_i (\|\chi_i u\|_{L^q(B_i)} + \|\chi_i f\|_{L^q(B_i)}) \\ &\le \underbrace{C(\epsilon)m}_{=:C}(\|u\|_{L^q(M)} + \|f\|_{L^q(M)}).\end{aligned}$$

The imbedding of Remark 20 is treated analogously. ☐

At the end we give a lemma which shows that solutions of the Euler-Lagrange equations have a maximum:

Lemma 22. *Let (M, g) be a manifold of bounded geometry. Let $v \in H_1^2$ be a solution of $Lv = c\rho^{\alpha p} v^{p-1}$ with $\|\rho^\alpha v\|_p = 1$ for $p < p_{crit}$. Then, $\limsup_{|x|\to\infty} v(x) = 0$, in particular v has a maximum.*

Proof. Assume, that there exist a constant $V > 0$ and a sequence $x_i \in M$ with $v(x_i) \ge V$ and $dist(x_i, p) \to \infty$ with $dist(x_i, x_j) > 2\epsilon$ for fixed $p \in M$. We set $B_i = B_\epsilon(x_i)$. Then, the interior L^p-estimates from above give $\|v\|_{H_2^q(B_i)} \le C_\epsilon(q)(\|v\|_{L^q(B_i)} + \|\rho^{\alpha p} v^{p-1}\|_{L^q(B_i)})$. Moreover, the Sobolev embedding in Theorem 19 shows that $v \in L^p$. From $\rho^\alpha v \in L^p$, $Lv = c\rho^{\alpha p} v^{p-1}$ and $0 \le \rho \le 1$, we obtain $Lv \in L^{q_1}$ with $q_1 = \frac{p}{p-1}$. The Schauder estimate above gives $v \in H_2^{q_1}(B_i)$ with $\|v\|_{H_2^{q_1}(B_i)} \le CC_\epsilon$. Then the Sobolev embedding give $\|v\|_{L^{p_1}(B_i)} \le C_\epsilon CC'$ with $p_1 = \frac{nq_1}{n-q_1}$ and where C' is the constant appearing in the corresponding Sobolev embedding. By a bootstrap argument we obtain for large q that $\|v\|_{H_2^q(B_i)} \le K(q)$ where the constant $K(q)$ depends on q but not on i. This bootstrap works since $p < p_{crit}$. Thus, with Remark 20.ii we get that $\|v\|_{C^{0,\alpha}(B_i)} \le c_\alpha$ where c_α is

independent of i.
From Theorem 19 we get from $v \in H_1^2$ that $v \in L^p$. Thus,

$$\infty > \|v\|_p \geq \sum_i \|v\|_{L^p(B_\delta(x_i))} \geq K \sum_i \min_{x \in B_\delta(x_i)} v(x)$$

where $K^p = \inf \mathrm{vol}(B_\delta(x_i))$ and $\delta \leq \epsilon$. Thus, $\min_{x \in B_\delta(x_i)} v(x) \to 0$ as $i \to \infty$. But we know that on each $B_\delta(x_i)$ we have $|v(x) - v(y)| \leq c_\alpha |x-y|^\alpha \leq c_\alpha \delta^\alpha$. Thus in the limit for $i \to \infty$ we get $V \leq c_\alpha \delta^\alpha$. Choosing δ small enough we have a contradiction. Thus, $\limsup_{|x| \to \infty} v(x) = 0$. $\square$

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